KLEIN PARADOX AND SCATTERING THEORY FOR THE SEMI-CLASSICAL DIRAC EQUATION

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Abstract. We study the Klein paradox for the semi-classical Dirac operator on $\mathbb{R}$ with potentials having constant limits, not necessarily the same at infinity. Using the complex WKB method, the time-independent scattering theory in terms of incoming and outgoing plane wave solutions is established. The corresponding scattering matrix is unitary. We obtain an asymptotic expansion, with respect to the semi-classical parameter h, of the scattering matrix in the cases of the Klein paradox, the total transmission and the total reflection. Finally, we treat the scattering problem in the zero mass case.

Keywords: Semi-classical Dirac operator - Scattering matrix - Klein paradox - Complex WKB method.

Mathematics classification: 81Q05 - 47A40 - 34L40 - 34E20 - 34M60.

1. Introduction

In mathematics and physics, the scattering theory is a framework for studying and understanding the scattering of waves and particles. The scattering matrix for the one-dimensional Dirac operator H is closely related to the transition probability of particles through a potential. However, if the potential does not vanish at infinity, a Klein paradox might occur. The latter is of great historical importance in order to justify the existence of the antiparticle of an electron (the positron) and explaining qualitatively the pair creation process in the collision of particle beam with strongly repulsive electric field. The explanation of this Klein paradox usually resorted to the concept of "hole" in the "negative-energy electron sea". For more physical interpretations we refer to Klein [14], Sauter [21], Bjorken-Drell [2], Sakurai [20], Thaller [23] and Calogeracos-Dombey [3] for the history of the Klein paradox. This paradox appears also for the Klein-Gordon equation, here no concept of "hole" is needed (see Winter [25] and Ni-Zhou-Yan [16] for a constant potential at infinity and Bachelot [1] for an electrostatic potential having different asymptotics at infinity). The comparison between the Klein paradox for this two equations has been discussed in [25, Part C]. A Klein paradox phenomenon occurs also in quantum field theory (see Hund [13] and Manogue [15]). It is clear that this paradox cannot appear for Schrödinger operators.

For a scalar potential having real limits $V^{\pm}$ at $\pm\infty$, the Klein paradox of the Dirac equation occurs if $V^{+}-V^{-} > 2mc^2$. In this case the higher part of $\sigma(H)$ intersects its lower part. If the energy E is in this intersection, for a wave-packet which comes from the left and moves towards the potential, a part of it is reflected, another part being transmitted. The transmitted part moves to the right and behaves like a solution with negative energy. Ruijsenaars-Bongaarts [19] (see also Thaller [23]) have mathematically treated the Klein paradox and the scattering

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theory for the Dirac equation with one-dimensional potentials constant outside a compact set. They have established the connection between time-dependent and time-independent scattering theory in terms of incoming and outgoing plane wave solutions. The exact calculus of the scattering matrix for one-dimensional Dirac operator is only known for a few number of explicit potentials (see Klein [14] for a rectangular step potential and Flügge [6] for the potential $V(x) = \tanh(x)$). Nevertheless, we are neither aware of works dealing with the asymptotic expansion of the scattering matrix, with respect to the semi-classical parameter h.

For one-dimensional Schrödinger operators, there are several approaches which have been developed dealing with the computation of the transmission coefficient through a barrier. Ecalle [5] and Voros [24] have developed the so-called complex WKB analysis which gives approximations in the complex plane of the solutions of a Schrödinger equation. This approach is used in a new formalism by Grigis for the Hill's equation [11]. This method is also used by C. Gérard-Grigis [10] to calculate the eigenvalues near a potential barrier and by Ramond [18] for scattering problems. For references and a historical discussion, we refer to Ramond [18]. The complex WKB method has been extended to a class of Schrödinger systems by Fujiié-Lasser-Nédelec [9].

The purpose of this paper is to give an asymptotic expansion, with respect to the semi-classical parameter h, of the coefficients of the scattering matrix for the one-dimensional Dirac operator with potentials having different limits at infinity. We establish the exponential decay of the transmission coefficient in the Klein paradox case (cf. Theorem 2.1 below). We calculate the coefficients of the scattering matrix in terms of incoming and outgoing solutions. Therefore, we use the complex WKB analysis to construct solutions of the Dirac equation. The usefulness of this analysis is that it provides, rather than approximate solutions with error bounds, solutions in the complex plane with a complete asymptotic expansion with respect to the semi-classical parameter h, with a priory estimates on the coefficients.

The paper is organized as follows. In the next section, we introduce the perturbed Dirac operator on $\mathbb{R}$, study the time-independent scattering theory and state our main results. In Section 3, we develop the complex WKB method and show a complete asymptotic expansion of the coefficients. In Section 4, the existence of incoming and outgoing Jost solutions is proved. In Section 5, we analyze the semi-classical behavior of the scattering matrix in the Klein paradox case. The total transmission over a potential barrier and the total reflection are studied in Section 6 and Section 7. Finally, in Section 8, we study the Klein paradox for the zero mass case.

2. Assumptions and results

We consider the self-adjoint Hamiltonian $H = H_0 + V$, where H_0 is the semi-classical Dirac operator on $\mathbb{R}$:

$$H_0 = -ihc\alpha\frac{d}{dx} + mc^2\beta, \tag{2.1}$$

with domain $D(H_0) = H^1(\mathbb{R}) \otimes \mathbb{C}^2 \subset \mathcal{H} = L^2(\mathbb{R}) \otimes \mathbb{C}^2$, where $h \searrow 0$ is the semi-classical parameter, $m \geq 0$ is the mass of the Dirac particle and c is the celerity of the light. The coefficients α, β are the 2×2 Pauli matrices satisfying the anti-commutation relation

$$\alpha\beta + \beta\alpha = 0,$$

and $\alpha^2 = \beta^2 = I_2$, where I_2 is the 2×2 identity matrix.

The operator V is the multiplication by VI_2, where V is a smooth electrostatic potential satisfying:

(A): *The function* V *is real on the real axis, analytic in the sector*

$$\mathcal{S} = \{x \in \mathbb{C},\ |\mathrm{Im}\ x| < \epsilon|\mathrm{Re}\ x| + \eta\},$$

for some $\epsilon, \eta > 0$, *and satisfies the following estimates:*

$$|(V(x) - V^{\pm})| = O(\langle x\rangle^{-\delta}) \quad for\ \mathrm{Re}\ (x) \longrightarrow \pm\infty \ \ in\ \ \mathcal{S}. \tag{2.2}$$

Here, $\langle x\rangle = (1+|x|^2)^{\frac{1}{2}}$, $\delta > 1$ *and* $V^- < V^+$.

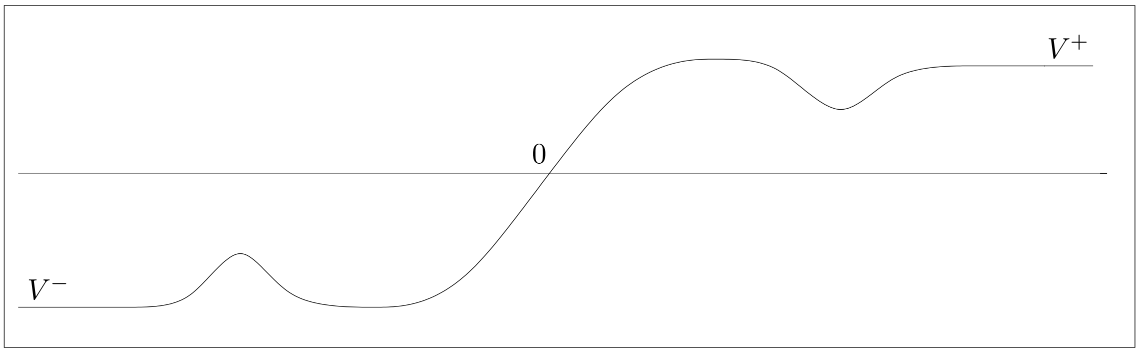

Fig. 1. The potential V

The spectrum of the free Dirac operator H_0 is $]-\infty, -mc^2] \cup [mc^2, +\infty[$ and it is purely absolutely continuous. Under assumption (A) the operator $H = H_0 + V$ is a self-adjoint operator and has essential spectrum (see Appendix A):

$$\sigma_{ess}(H) \quad = \quad]-\infty, -mc^2 + V^+] \cup [mc^2 + V^-, +\infty[. \tag{2.3}$$

There are several representations of the matrices α, β. For example, Hiller [12] used $\alpha = \sigma_2,\ \beta = \sigma_3$, Nogami and Toyoma [17] used $\alpha = \sigma_2,\ \beta = \sigma_1$, where $\sigma_j,\ j = 1, 2, 3$, are the standard representation for Dirac-Pauli matrices. Most calculations with Dirac matrices can be done without referring to a particular representation (see Thaller [23, Appendix 1A]). Here, we choose the $1+1$ dimensional representation of the Dirac matrices

$$\alpha = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \ \beta = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.4}$$

The solutions of

$$Hu = \left(-ihc\sigma_1 \frac{d}{dx} + mc^2\sigma_3 + V(x)I_2\right) u = Eu, \qquad E \in \mathbb{R}, \tag{2.5}$$

should behave as $x \longrightarrow \pm\infty$ like

$$a^+_{\pm}(E,h)\exp(+\frac{1}{hc}(m^2c^4 - (V^{\pm} - E)^2)^{1/2}x) + a^-_{\pm}(E,h)\exp(-\frac{1}{hc}(m^2c^4 - (V^{\pm} - E)^2)^{1/2}x).$$

Here, the square root $(\cdot)^{\frac{1}{2}}$ is to be defined more precisely according to the sign of $m^2c^4 - (V^{\pm} - E)^2$.

In the following, we will use these intervals on the E-axis:

I. $V^+ + mc^2 < E$,

II. $\max(mc^2 + V^-, V^+ - mc^2) < E < V^+ + mc^2$,

III. $V^- + mc^2 < E < V^+ - mc^2 \quad$ if $\quad V^+ - V^- > 2mc^2$,

IV. $V^- - mc^2 < E < \min(V^- + mc^2, V^+ - mc^2)$,

V. $E < V^- - mc^2$.

If $V^+ - V^- > 2mc^2$, the different regions are represented in the following figure:

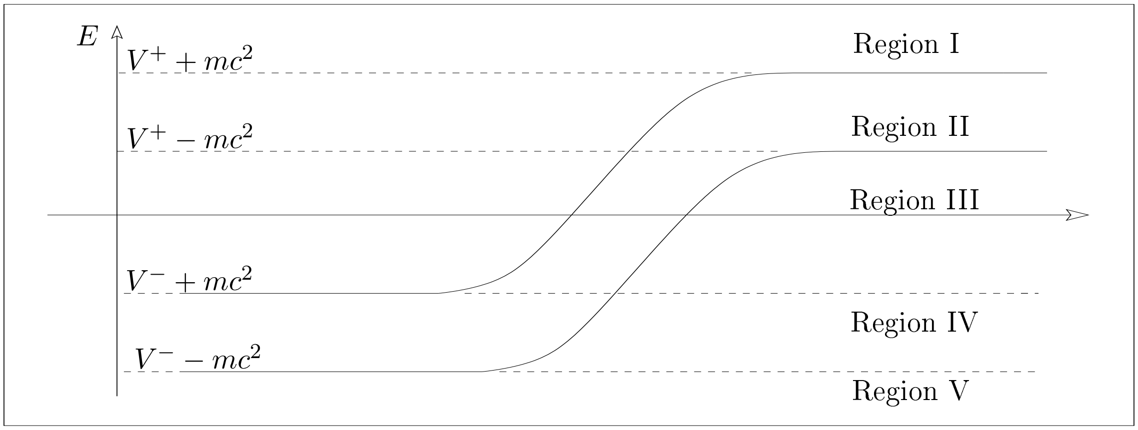

Fig. 2. Different regions on the E-axis

We study the semi-classical behavior of the scattering matrix for the different values of the energy E. Let us describe now the time-independent scattering problem briefly. For $E \in$ I, III or V, the four Jost solutions $\omega_{\text{in}}^{\pm}$, $\omega_{\text{out}}^{\pm}$ (see Theorem 4.1) are the solutions of (2.5) which behave exactly as

$$\omega_{\text{in}}^{\pm} \sim \exp\{\mp \frac{i}{hc}\Phi(E - V^{\pm})x\} \begin{pmatrix} A(E - V^{\pm}) \\ \mp A(E - V^{\pm})^{-1} \end{pmatrix} \quad \text{as } x \longrightarrow \pm\infty, \tag{2.6}$$

$$\omega_{\text{out}}^{\pm} \sim \exp\{\pm \frac{i}{hc}\Phi(E - V^{\pm})x\} \begin{pmatrix} A(E - V^{\pm}) \\ \pm A(E - V^{\pm})^{-1} \end{pmatrix} \quad \text{as } x \longrightarrow \pm\infty, \tag{2.7}$$

with $\Phi(E) = \text{sgn}(E)\sqrt{E^2 - m^2c^4}$, $A(E) = \sqrt[4]{\frac{E+mc^2}{E-mc^2}}$ and $\text{sgn}(E) = \frac{E}{|E|}$ for $E \notin [-mc^2, mc^2]$. Analogous definitions of Jost solutions can be found in the works of Ruijsenaars-Bongaarts [19] and Thaller [23] for one-dimensional step potentials. In this paper, we denote $\sqrt{x}$, $\sqrt[4]{x}$ the positive determination of $x \in \mathbb{R}^+ \longrightarrow (x)^{\frac{1}{2}}$, $(x)^{\frac{1}{4}}$ respectively.

The ordinary scattering problem is the following: what are the components of a solution u of the Dirac equation (2.5) in the basis $(\omega_{\text{out}}^+, \omega_{\text{out}}^-)$ of the outgoing Jost solutions, knowing its component in the basis $(\omega_{\text{in}}^-, \omega_{\text{in}}^+)$ of the incoming Jost solutions. The 2×2 matrix relating these coefficients is called the scattering matrix and we will denote it by

$$\mathbb{S} = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}.$$

Precisely, if we take u a solution of (2.5),

$$u = a_{\text{in}}\omega_{\text{in}}^- + b_{\text{in}}\omega_{\text{in}}^+ = a_{\text{out}}\omega_{\text{out}}^+ + b_{\text{out}}\omega_{\text{out}}^-,$$

the scattering matrix is such that

$$\mathbb{S}\begin{pmatrix} a_{\text{in}} \\ b_{\text{in}} \end{pmatrix} = \begin{pmatrix} a_{\text{out}} \\ b_{\text{out}} \end{pmatrix},$$

which is equivalent to

$$(\omega_{\text{in}}^-, \omega_{\text{in}}^+) = (\omega_{\text{out}}^+, \omega_{\text{out}}^-)\mathbb{S}. \tag{2.8}$$

Since V is real on the real axis, we have (see (2.4))

$$\overline{\omega_{\text{in}}^{\pm}} = \beta\omega_{\text{out}}^{\pm}. \tag{2.9}$$

We also have the following relations between the coefficients of $\mathbb{S}(E,h)$:

$$s_{11}(E,h) = s_{22}(E,h) \quad \text{and} \quad s_{12}(E,h) = -\overline{s_{21}}(E,h)\frac{s_{11}(E,h)}{\overline{s_{11}}(E,h)}, \tag{2.10}$$

so that s_{11} and s_{12} determine completely the scattering matrix.

The reflection and transmission coefficients $R(E,h)$ and $T(E,h)$ are, by definition, the square of the modulus of the coefficients s_{21} and s_{11} respectively. They correspond to the probability for a purely incoming-from-the-left particle to be reflected to the left or transmitted to the right. Using (2.9), (2.10) and calculating the determinant of (2.8), we have the well-known relation $R(E,h)+T(E,h)=1$ and, the scattering matrix $\mathbb{S}(E,h)$ is unitary.

To calculate the scattering matrix $\mathbb{S}(E,h)$ we will use the transfer matrix $\mathbb{T}$, which is defined by

$$(\omega_{\text{in}}^{-},\ \omega_{\text{out}}^{-}) = (\omega_{\text{out}}^{+},\ \omega_{\text{in}}^{+})\mathbb{T}.$$

The determinant of this matrix is equal to 1 since the two Wronskians $\mathcal{W}(\omega_{\text{in}}^{-},\ \omega_{\text{out}}^{-})$ and $\mathcal{W}(\omega_{\text{out}}^{+},\ \omega_{\text{in}}^{+})$ are equal to -2 (see Definition 3.4). Using the relation (2.9), we obtain that $\mathbb{T}$ is determined by two coefficients:

$$\mathbb{T} = \begin{pmatrix} t(E,h) & r(E,h) \\ \overline{r}(E,h) & \overline{t}(E,h) \end{pmatrix}. \tag{2.11}$$

Moreover, using that $\det(\mathbb{T})=1$, we obtain

$$|t(E,h)|^2 - |r(E,h)|^2 = 1. \tag{2.12}$$

Consequently, we can write the scattering matrix in terms of the coefficients of the transfer matrix $\mathbb{T}$:

$$\mathbb{S} = \frac{1}{\overline{t}(E,h)}\begin{pmatrix} 1 & -r(E,h) \\ \overline{r}(E,h) & 1 \end{pmatrix}. \tag{2.13}$$

We will use WKB approaches to describe the amplitude of the coefficients of the scattering matrix for $h \searrow 0$. For these, let us introduce the following definition.

Definition 2.1. *(See Sjöstrand [22]) A function $f(z,h)$ defined in $U\times]0,h_0[$, where U is an open set in $\mathbb{C}$ and $h_0>0$, is called a classical analytic symbol (CAS) of order $m\in\mathbb{N}$ in h if f is an analytic function of $z\in U$ and if there exists a sequence $(a_j(z))$ of analytic functions in U such that*

- *For all compact set $K\subset U$, there exists $C>0$ such that, for all $z\in K$, one has*
$$|a_j(z)| \le C^{j+1}j^j.$$
- *The function $f(z,h)$ admits the series $\sum_{0\le j\le+\infty} a_j(z)h^{m+j}$ as asymptotic expansion as h goes to zero in the following sense. For any $C_1>C$, we have*
$$f(z,h) - \sum_{0\le j\le h^{-1}/eC_1} a_j(z)h^{m+j} = O(e^{-\rho/h}),$$
for some $\rho>0$ and all $z\in K$

The main theorem concerning the Klein paradox case for $m > 0$ (i.e. for the energy level $E \in \mathrm{III}$) is the following:

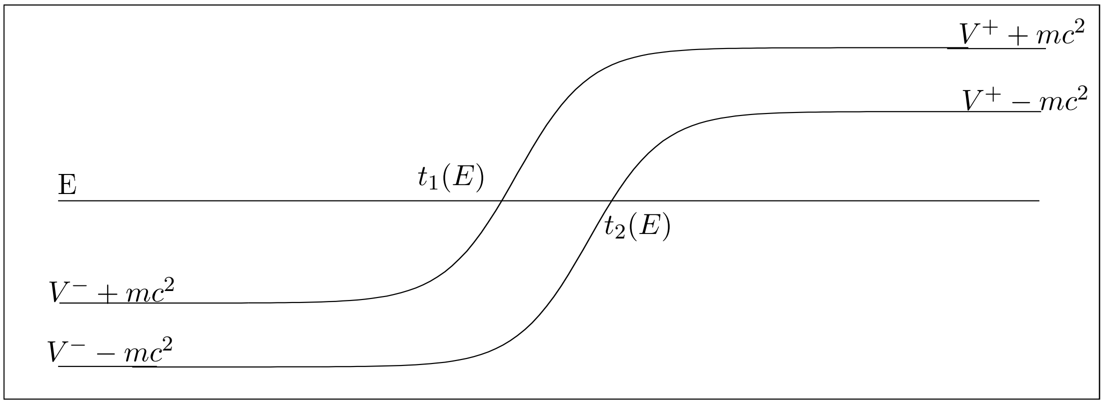

Fig. 3. Graph of $V(x) + mc^2$ and $V(x) - mc^2$

Theorem 2.1. **[Klein paradox]** *Let V be a potential satisfying assumption (A), $E \in \mathrm{III}$ and $m > 0$. Suppose that there are only two simple zeros $t_1(E) < t_2(E)$ of $m^2c^4 - (V(x) - E)^2$ (see Fig. 3). Then there exists three classical analytic symbols $\phi_1(h)$, $\phi_2(h)$ and $\phi_3(h)$ of non-negative order such that:*

$$s_{11} = s_{22} = (1 + h\phi_1(h)) \exp\{-S(E)/h\} \exp\{iT(E)/h\}, \tag{2.14}$$

$$s_{21} = (i + h\phi_2(h)) \exp\left\{\frac{2i}{h}\left(t_1(E)\sqrt{E^-} + \int_{-\infty}^{t_1(E)} Q^-(t, E)dt\right)\right\}, \tag{2.15}$$

$$s_{12} = (i + h\phi_3(h)) \exp\left\{\frac{2i}{h}\left(t_2(E)\sqrt{E^+} - \int_{t_2(E)}^{+\infty} Q^+(t, E)dt\right)\right\}, \tag{2.16}$$

where $S(E)$ is the classical action between the two turning points $t_1(E)$ and $t_2(E)$

$$S(E) = \int_{t_1(E)}^{t_2(E)} \sqrt{\frac{m^2c^4 - (V(t) - E)^2}{c^2}} dt.$$

Moreover

$$Q^-(t, E) = \sqrt{\frac{-m^2c^4 + (V(t) - E)^2}{c^2}} - \sqrt{E^-}, \quad \text{for } t < t_1(E),$$

$$Q^+(t, E) = \sqrt{\frac{-m^2c^4 + (V(t) - E)^2}{c^2}} - \sqrt{E^+}, \quad \text{for } t > t_2(E),$$

and

$$T(E) = \int_{-\infty}^{t_1(E)} Q^-(t, E)dt - \int_{t_2(E)}^{+\infty} Q^+(t, E)dt + t_1(E)\sqrt{E^-} + t_2(E)\sqrt{E^+},$$

where

$$E^\pm = \frac{-m^2c^4 + (V^\pm - E)^2}{c^2}.$$

We remark that this scattering matrix behaves like in the case of the Schrödinger operator with a barrier potential. In particular the term $e^{-S(E)/h}$ which decays exponentially, can be viewed as a tunneling effect (see Ramond [18, Theorem 1]).

In the zero mass case, we have the following theorem:

Theorem 2.2. [**Zero mass case**] *Let V be a potential satisfying assumption (A), $E \in$ III and $m = 0$. Suppose that there is only a simple zero $t_0(E)$ of $V(x) - E$. Then, there is a classical analytic symbol $\phi(h)$ such that:*

$$s_{11} = s_{22} = (1 + h\phi(h)) \exp\{iT_0(E)/h\}, \tag{2.17}$$

$$s_{21} = O(h), \tag{2.18}$$

$$s_{12} = O(h). \tag{2.19}$$

Here,

$$\begin{aligned} T_0(E) &= T(E)\Big|_{m=0} \\ &= \frac{1}{c}\left(\int_{-\infty}^{t_0(E)} (-V(t) + V^-)dt - \int_{t_0(E)}^{+\infty} (V(t) - V^+)dt + t_0(E)(V^+ - V^-)\right). \end{aligned}$$

Remark 2.1. *We can not permute the limits of the scattering matrix $\mathbb{S}$ as $m \to 0$ and $h \to 0$. Indeed, if we take the limits of s_{12} in (2.16) and (2.19), we obtain*

$$\lim_{m\to 0} \lim_{h\to 0} |s_{12}| = 1, \qquad \lim_{h\to 0} \lim_{m\to 0} |s_{12}| = 0.$$

Now, we come back to the non-zero mass case and we treat reflection and transmission cases (see Sections 7, 6).

If we take the energy level $E \in$II, there are two Jost solutions ω_{in}^-, ω_{out}^- satisfying (2.6) and (2.7) for $x \longrightarrow -\infty$ and there does not exist an oscillating solution for $x \longrightarrow +\infty$. Instead, as $x \to +\infty$, there exists an exponentially decaying solution and an exponentially growing solution. Since the last function doesn't represent a physical state we limit ourself to the one dimensional space generated by the decaying solution ω_d^+ (unique up to a constant). This function satisfies (see Theorem 4.1):

$$\omega_d^+ \sim \exp\{-\frac{1}{hc}\sqrt{m^2c^4 - (V^+ - E)^2}x\} \begin{pmatrix} -i\sqrt[4]{\frac{mc^2+E-V^+}{mc^2-E+V^+}} \\ \sqrt[4]{\frac{mc^2-E+V^+}{mc^2+E-V^+}} \end{pmatrix} \quad \text{as } x \longrightarrow +\infty. \tag{2.20}$$

In this case we have

Theorem 2.3. [**Total reflection**] *Let V be a potential satisfying assumption (A), $E \in$ II and $m > 0$. Suppose that there is only a simple zero $t_1(E)$ of $m^2c^4 - (V(x) - E)^2$. Then the vector space of the solutions of $(H - E)u = 0$ with u bounded is a one dimensional space generated by*

$$u = \omega_{\text{in}}^- + \alpha_{\text{out}}^- \omega_{\text{out}}^-, \quad \textit{with}$$

$$\alpha_{\text{out}}^{-}=-i(1+h\phi_1(h))\exp\left\{\frac{2i}{h}\left(\int_{-\infty}^{t_1(E)}Q^{-}(t,E)dt+\sqrt{E^{-}}t_1(E)\right)\right\}. \tag{2.21}$$

Moreover

$$u=\beta_d^{+}\omega_d^{+},\qquad \textit{with}$$

$$\begin{aligned}\beta_d^{+} &= e^{i\pi/4}(1+h\phi_2(h))\times\\ &\exp\left\{\frac{1}{h}\left(\int_{+\infty}^{t_1(E)}Q_{-}^{+}(t,E)dt+\sqrt{-E^{+}}t_1(E)+i\int_{-\infty}^{t_1(E)}Q^{-}(t,E)dt+i\sqrt{E^{-}}t_1(E)\right)\right\}.\end{aligned}$$

Here

$$Q_{-}^{+}(t,E)=\sqrt{\frac{m^2c^4-(V(t)-E)^2}{c^2}}-\sqrt{-E^{+}},$$

$Q^{-}(t,E)$ and $E^{\pm}$ are the functions of Theorem 2.1 and $\phi_j(h)$, $j=1,2$ are classical analytic symbols of non-negative order.

For $E\in$ IV, there is also total reflection cases which can treated similarly to the previous theorem. As in [19], there is also a scattering interpretation of the previous theorem. Since we work in a one-dimensional space, the scattering matrix is now a scalar.

Remark 2.2. [**Scattering interpretation**] *We call $u^{\text{in}}=\omega_{\text{in}}^{-}+\alpha_{\text{out}}^{-}\omega_{\text{out}}^{-}$ the "incoming" solution. In the same way, there exists a unique bounded solution*

$$u^{\text{out}}=\omega_{\text{out}}^{-}+\alpha_{\text{in}}^{-}\omega_{\text{in}}^{-},$$

which is called the "outgoing" solution.

If u is a bounded solution of $(H-E)u=0$ (i.e. $u=Au^{\text{in}}$) then $u=Bu^{\text{out}}$. The scattering matrix $\mathbb{S}$ is defined by

$$B=\mathbb{S}A.$$

From (2.21), we have

$$\mathbb{S}=\alpha_{\text{out}}^{-}=-i(1+h\phi(h))\exp\left\{\frac{2i}{h}\left(\int_{-\infty}^{t_1(E)}Q^{-}(t,E)dt+\sqrt{E^{-}}t_1(E)\right)\right\}.$$

For $E\in$ I or V, a total transmission phenomena occur:

Theorem 2.4. [**Total transmission**] *Let V be a potential satisfying assumption (A), $E\in$ I, $m\geq 0$ and $m^2c^4-(V(x)-E)^2\neq 0$. Then there are a classical analytic symbol $\phi(h)$ and positive constant C such that:*

$$s_{11} = s_{22}=(1+h\phi(h))\exp\{i\widetilde{T}(E)/h\}, \tag{2.22}$$

$$s_{21} = O(e^{-C/h})\quad \textit{and}\quad s_{12}=O(e^{-C/h}), \tag{2.23}$$

where

$$\widetilde{T}(E)=\int_{-\infty}^{0}Q^{-}(t,E)dt+\int_{0}^{+\infty}Q^{+}(t,E)dt,$$

and $Q^{-}(t,E)$, $Q^{+}(t,E)$ are the functions of Theorem 2.1 defined here for any $t\in\mathbb{R}$.

We can calculate the scattering matrix for $E\in$ V in the same way of $E\in$ I. We remark that the behavior of the incoming and outgoing Jost solutions exchanges between these two cases. This is in agreement with the physical interpretation (see [23, p.121]).

3. Complex WKB solutions

We wish to find a representation formula for the solutions of (2.5), from which it is possible to deduce the asymptotic expansion in h. The method is known as complex WKB method. See [18] [10], [7], [8], [9] for constructions of solutions of the Schrödinger equation.

In a complex domain $\mathcal{S}$, we study the Dirac system (2.5) which is of the form

$$(H-E)u(x) = \begin{pmatrix} mc^2+V(x)-E & -ihc\frac{d}{dx} \\ -ihc\frac{d}{dx} & -mc^2+V(x)-E \end{pmatrix} u(x) = 0, \tag{3.1}$$

or equivalently

$$\frac{h}{i}\frac{d}{dx}v(x) = \begin{pmatrix} 0 & g_+(x) \\ -g_-(x) & 0 \end{pmatrix} v(x), \tag{3.2}$$

where $v(x) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} u(x) = M^{-1}u$, and the functions

$$g_\pm(x) = \frac{-mc^2 \mp (V(x)-E)}{c},$$

are holomorphic in $\mathcal{S}$. The following considerations will lead to the construction of complex WKB solutions for Dirac system.

3.1. **Formal construction.**

First, we introduce a new complex coordinate

$$z(x) = z(x,x_0) = \int_{\gamma(x_0,x)} (g_+(t)g_-(t))^{\frac{1}{2}}dt = \int_{x_0}^{x} (g_+(t)g_-(t))^{\frac{1}{2}}dt, \quad x_0 \in D. \tag{3.3}$$

One of our tasks will be of course to choose the simply connected subset D of $\mathcal{S}$ such that $t \longrightarrow (g_+(t)g_-(t))^{\frac{1}{2}}$ is well-defined, but let's work formally for a while. The $\gamma(x_0,x)$ is any path in D beginning at x_0 and ending at x.

Definition 3.1. *The zeros of the function*

$$g_+(x)g_-(x) = \frac{m^2c^4-(V(x)-E)^2}{c^2},$$

are called the turning points of the system (3.2).

Definition 3.2. *For x fixed in D, the set*

$$\left\{ y \in D,\ Re \int_x^y (g_+(t)g_-(t))^{\frac{1}{2}}dt = 0 \right\}$$

is called the Stokes line passing through x.

We look for solutions of the form $e^{\pm\frac{z}{h}}\widetilde{w}_\pm(z)$. We note that due to the possible presence of such turning points, the square root in the definition of $z(x)$ might be defined only locally. By formal calculations, the amplitude vector $\widetilde{w}_\pm(z)$ has to satisfy

$$\frac{h}{i}\frac{d}{dz}\widetilde{w}_\pm(z) = \begin{pmatrix} \pm i & H(z)^{-2} \\ -H(z)^2 & \pm i \end{pmatrix} \widetilde{w}_\pm(z). \tag{3.4}$$

The function $H(z(x))$ is given by

$$H(z(x)) = \left(\frac{g_-(x)}{g_+(x)}\right)^{1/4} = \left(\frac{-mc^2 + (V(x) - E)}{-mc^2 - (V(x) - E)}\right)^{1/4}, \tag{3.5}$$

for $z(x)$ in an open simply-connected domain of the z-plane, where $z \longrightarrow H(z)$ is well-defined and analytic.

In order to obtain a decomposition with respect to image and kernel of the previous system, we conjugate by

$$P_\pm(z) = \frac{1}{2}\begin{pmatrix} H(z) & \pm iH(z)^{-1} \\ H(z) & \mp iH(z)^{-1} \end{pmatrix}, \quad P_\pm^{-1}(z) = \begin{pmatrix} H(z)^{-1} & H(z)^{-1} \\ \mp iH(z) & \pm iH(z) \end{pmatrix},$$

and obtain a system for $w_\pm(z) = P_\pm(z)\widetilde{w}_\pm(z)$,

$$\frac{d}{dz}w_\pm(z) = \begin{pmatrix} 0 & \frac{H'(z)}{H(z)} \\ \frac{H'(z)}{H(z)} & \mp\frac{2}{h} \end{pmatrix} w_\pm(z), \tag{3.6}$$

where $H'(z)$ is shorthand for $\frac{d}{dz}H(z)$. The series ansatz

$$w_\pm(z) = \sum_{n\geq 0}\begin{pmatrix} w_{2n,\pm}(z) \\ w_{2n+1,\pm}(z) \end{pmatrix}, \tag{3.7}$$

with $w_{0,\pm} = 1$ and, for $n \geq 1$, the recurrence equations

$$\left(\frac{d}{dz} \pm \frac{2}{h}\right) w_{2n+1,\pm}(z) = \frac{H'(z)}{H(z)} w_{2n,\pm}(z), \tag{3.8}$$

$$\frac{d}{dz} w_{2n+2,\pm}(z) = \frac{H'(z)}{H(z)} w_{2n+1,\pm}(z), \tag{3.9}$$

give us a formal solution up to some additive constants. The solutions are fixed by setting

$$w_{n,\pm}(\widetilde{z}) = 0, \quad n \geq 1,$$

at a base point $\widetilde{z} = z(\widetilde{x})$ where $\widetilde{x} \in D$ is not a turning point. We note that the previous equations for $w_{n,\pm}$ are similar to the ones obtained by a complex WKB construction for scalar Schrödinger equations. See for example the works of C. Gérard and Grigis [10] or Ramond [18].

Let $\Omega = \Omega(E)$ be a simply connected subset of D which does not contain any turning point. Then the function $z = z(x)$ is conformal from Ω onto $z(\Omega)$. Assume that $\widetilde{z} \in z(\Omega)$. If $\Gamma_\pm(\widetilde{z}, z)$ denotes a path of finite length in $z(\Omega)$ connecting $\widetilde{z}$ and $z \in z(\Omega)$, we can formally rewrite the above differential equations for $n \geq 0$ as

$$\begin{aligned} w_{2n+1,\pm}(z) &= \int_{\Gamma_\pm(\widetilde{z},z)} \exp(\pm\frac{2}{h}(\zeta - z))\frac{H'(\zeta)}{H(\zeta)} w_{2n,\pm}(\zeta)d\zeta, \\ w_{2n+2,\pm}(z) &= \int_{\Gamma_\pm(\widetilde{z},z)} \frac{H'(\zeta)}{H(\zeta)} w_{2n+1,\pm}(\zeta)d\zeta, \end{aligned}$$

or after iterated integrations, as

$$w_{2n+1,\pm}(z)=\int_{\Gamma_\pm(\widetilde{z},z)}\int_{\Gamma_\pm(\widetilde{z},\zeta_{2n+1})}\cdots\int_{\Gamma_\pm(\widetilde{z},\zeta_2)}\exp\left(\pm\frac{2}{h}(\zeta_1-\zeta_2+\cdots+\zeta_{2n+1}-z)\right)\times$$
$$\times\quad\frac{H'(\zeta_1)}{H(\zeta_1)}\cdots\frac{H'(\zeta_{2n+1})}{H(\zeta_{2n+1})}\,d\zeta_1\cdots d\zeta_{2n+1},$$
$$w_{2n+2,\pm}(z)=\int_{\Gamma_\pm(\widetilde{z},z)}\int_{\Gamma_\pm(\widetilde{z},\zeta_{2n+2})}\cdots\int_{\Gamma_\pm(\widetilde{z},\zeta_2)}\exp\left(\pm\frac{2}{h}(\zeta_1-\zeta_2+\cdots-\zeta_{2n+2})\right)\times$$
$$\times\quad\frac{H'(\zeta_1)}{H(\zeta_1)}\cdots\frac{H'(\zeta_{2n+2})}{H(\zeta_{2n+2})}\,d\zeta_1\cdots d\zeta_{2n+2}.$$

3.2. **Convergence, h-dependence and Wronskians.** We now give to the preceding formal construction some mathematical meaning in simply connected, turning point-free compact sets $\Omega\subset D$.

Lemma 3.1. *For any fixed $h>0$, the series (3.7) converges uniformly in any compact subset of Ω, and*

$$w_\pm^{\text{even}}(x,h)=\sum_{n\ge 0}w_{2n,\pm}(z(x)),\quad w_\pm^{\text{odd}}(x,h)=\sum_{n\ge 0}w_{2n+1,\pm}(z(x)),\tag{3.10}$$

are holomorphic functions in Ω.

Proof. By assumption on Ω and on V, the functions $w_{n,\pm}$ are well-defined analytic functions in Ω. For compact subsets $K\subset\Omega$ and $\widetilde{z},z\in z(K)$ there exist positive constants $C_\pm^h(K)>0$, depending on the semi-classical parameter h and the compact K such that

$$\sup_{\zeta\in\Gamma_\pm(\widetilde{z},z)}\left|\exp(\pm\frac{2}{h}\zeta)\frac{H'(\zeta)}{H(\zeta)}\right|\le C_\pm^h(K).\tag{3.11}$$

If we denote the maximal length of the paths $\Gamma_\pm(\widetilde{z},\cdot)\subset K$ in the preceding iterated integrations by

$$L=\max_{\widetilde{z},z\in z(K)}\min_{\gamma(\widetilde{z},z)}|\gamma(\widetilde{z},z)|<\infty,$$

then

$$\sup_{z\in z(K)}|w_{n,\pm}(z)|\le\frac{C_\pm^h(K)^nL^n}{n!},\quad n\ge 0,$$

where the bound $\frac{L^n}{n!}$ comes from the volume of a simplex with length L. Then, the lemma follows. □

Thus, we have the uniform convergence of the series (3.7) for $w_\pm(z)$ and complex solutions

$$u_\pm(x)=e^{\pm\frac{z(x)}{h}}T_\pm(z(x))\begin{pmatrix}w_\pm^{\text{even}}(x)\\ w_\pm^{\text{odd}}(x)\end{pmatrix},\tag{3.12}$$

of the original problem (3.1) on any turning point-free set Ω, where

$$\begin{aligned}T_\pm(z)=MP_\pm^{-1}(z)\quad&=\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}H(z)^{-1}&H(z)^{-1}\\ \mp iH(z)&\pm iH(z)\end{pmatrix}\\&=\begin{pmatrix}\mp iH(z)&\pm iH(z)\\ H(z)^{-1}&H(z)^{-1}\end{pmatrix},\qquad z\in z(\Omega).\end{aligned}\tag{3.13}$$

We write these solutions $u_{\pm}(x)$ as

$$u_{\pm}(x; x_0, \widetilde{x}), \tag{3.14}$$

indicating the particular choice of the phase base point x_0, in (3.3), which defines the phase function $z(x) = z(x; x_0)$, and the choice of the amplitude base point $\widetilde{z} = z(\widetilde{x})$, which is the initial point of the path $\Gamma_{\pm}(\widetilde{z}, \cdot)$.

Definition 3.3. *For $\widetilde{x} \in \Omega$ fixed, we define $\Omega_{\pm} = \Omega_{\pm}(\widetilde{x})$ the set of all $x \in \Omega$ such that there exists a path $\Gamma_{\pm}(z(\widetilde{x}), z(x))$ along which $x \longrightarrow \pm\mathrm{Re}\ z(x)$ increases strictly.*

Proposition 3.1. *The functions $w_{n,\pm}$ are classical analytic symbols of order $[\frac{n+1}{2}]$ in $\Omega_{\pm}$. The functions $w_{\pm}^{even}(x,h)$ and $w_{\pm}^{odd}(x,h)$ given by the identities (3.10) are classical analytic symbols of order 0 and 1 respectively in $\Omega_{\pm}$. Moreover, we have for any $N \in \mathbb{N}$,*

$$\begin{aligned} w_{\pm}^{even}(x,h) - \sum_{n=0}^{N} w_{2n,\pm}(z(x)) &= O(h^{N+1}), \\ w_{\pm}^{odd}(x,h) - \sum_{n=0}^{N} w_{2n+1,\pm}(z(x)) &= O(h^{N+2}), \end{aligned}$$

uniformly in any compact subsets of $\Omega_{\pm}$. In particular,

$$w_{\pm}^{even}(x,h) = 1 + h\phi(h), \quad w_{\pm}^{odd}(x,h) = h\phi(h).$$

Here and in all this paper, $\phi(h)$ is a classical analytic symbol of non-negative order not necessarily the same in each expression.

The proof is just the same as that of [10, Prop. 1.2] and [9, Prop. 3.3]. The key point is the following: since the iterated integrations defining $w_{n,\pm}(z)$ contain terms of the form $\exp(\pm\frac{\zeta}{h})$, one has to make sure that $x \longmapsto \pm\mathrm{Re}\ (z(x))$ is a strictly increasing function along the path $\Gamma_{\pm}(\widetilde{z}, z)$. In other words, the paths $\Gamma_{\pm}(z(\widetilde{x}), z(x))$ have to intersect the Stokes lines, that is the level curves of $x \longmapsto \mathrm{Re}\ (z(x))$, transversally in a suitable direction.

Definition 3.4. *One defines the Wronskian of two $\mathbb{C}^2$-valued functions $u = (u_1, u_2)$, $v = (v_1, v_2)$ as*

$$\mathcal{W}(u,v) = u_1 v_2 - u_2 v_1.$$

Remark 3.1. *For two solutions u and v of the equation (2.5), the Wronskian $\mathcal{W}(u,v)$ doesn't depend on x and is zero if and only if u and v are proportional.*

If $w = \alpha u + \beta v$ with α, $\beta \in \mathbb{C}$, then

$$\alpha = \frac{\mathcal{W}(w,v)}{\mathcal{W}(u,v)}, \quad \beta = -\frac{\mathcal{W}(w,u)}{\mathcal{W}(u,v)}.$$

Elementary computations give the following complex Wronskian formulas for complex WKB solutions with different phase and amplitude base points in terms of $w_{\pm}^{\mathrm{even}}$ and $w_{\pm}^{\mathrm{odd}}$.

Lemma 3.2. *Let x_0 and y_0 be two points in $\Omega = \Omega(E)$. If, for given $\widetilde{x}$ and $\widetilde{y}$, the canonical sets $\Omega_{\pm}(\widetilde{x})$ and $\Omega_{\pm}(\widetilde{y})$ have a non-empty intersection, then for any $x \in \Omega_{\pm}(\widetilde{x}) \cap \Omega_{\pm}(\widetilde{y})$ one*

has

$$\text{(3.15)}\quad \mathcal{W}(u_\pm(x;x_0,\widetilde{x})\,,u_\pm(x;y_0,\widetilde{y})) = \pm 2i\exp\left(\pm\frac{1}{h}(z(x;x_0)+z(x;y_0))\right) \\ \times\quad \left(w_\pm^{\text{odd}}(x;x_0,\widetilde{x})w_\pm^{\text{even}}(x;y_0,\widetilde{y}) - w_\pm^{\text{even}}(x;x_0,\widetilde{x})w_\pm^{\text{odd}}(x;y_0,\widetilde{y})\right).$$

If, for given $\widetilde{x}$ and $\widetilde{y}$ the canonical sets $\Omega_\pm(\widetilde{x})$ and $\Omega_\mp(\widetilde{y})$ have a non-empty intersection, then for any $x \in \Omega_\pm(\widetilde{x}) \cap \Omega_\mp(\widetilde{y})$ one has

$$\text{(3.16)}\quad \mathcal{W}(u_\pm(x;x_0,\widetilde{x})\,,u_\mp(x;y_0,\widetilde{y})) = \pm 2i\exp\left(\pm\frac{1}{h}(z(x;x_0)-z(x;y_0))\right) \\ \times\quad \left(w_\pm^{\text{odd}}(x;x_0,\widetilde{x})w_\mp^{\text{odd}}(x;y_0,\widetilde{y}) - w_\pm^{\text{even}}(x;x_0,\widetilde{x})w_\mp^{\text{even}}(x;y_0,\widetilde{y})\right).$$

4. Jost solutions

The Jost solutions of $Hu = Eu$, are characterized by the behavior of the solutions at infinity. We construct here the Jost solutions copying the procedure described in Section 3, the new point here being that the solutions we seek are normalized at infinity. In all this section we will work in two unbounded, simply-connected domains $\Omega^-(E)$, $\Omega^+(E)$, where Re $(V(x)+mc^2) < E$, Re $(V(x)-mc^2) > E$ respectively and which coincide with $\mathcal{S}$ for Re x sufficiently large. The existence of such domains is of course an easy consequence of the behavior of V at infinity in $\mathcal{S}$ (see assumption (A)).

First we define the phase functions with base point at infinity,

$$\text{(4.1)}\qquad z(x,\pm\infty) = \int_{\pm\infty}^{x}\left(\frac{m^2c^4-(V(t)-E)^2}{c^2}\right)^{1/2} - \left(\frac{m^2c^4-(V^\pm-E)^2}{c^2}\right)^{1/2} dt \\ + \left(\frac{m^2c^4-(V^\pm-E)^2}{c^2}\right)^{1/2} x.$$

We also see that the integral converges absolutely, hence

$$z(x,\pm\infty) = \left(\frac{m^2c^4-(V^\pm-E)^2}{c^2}\right)^{1/2} x + o(1), \qquad (x \longrightarrow \pm\infty).$$

If the determination of the square root in $z(\cdot,\cdot)$ are the same, we get the following equalities

$$\text{(4.2)}\qquad \begin{aligned} z(t_1,\pm\infty) =& z(x,\pm\infty) - z(x,t_1) = z(t_2,\pm\infty) - z(t_2,t_1) \\ z(t_1,-\infty) - z(t_1,+\infty) =& z(x,-\infty) - z(x,+\infty) = z(t_2,-\infty) - z(t_2,+\infty), \end{aligned}$$

where $z(\cdot,\cdot)$ is defined in (3.3), (4.1) and $x,\ t_1,\ t_2 \in D$.

Next we define the amplitudes based at infinity. We will only define the amplitudes at $+\infty$ since the situation is similar at $-\infty$. As in Section 3 of [18], we choose infinite paths $\gamma_\pm(x)$ starting from infinity and ending at x, which are asymptotically like lines of the form $\{\mathrm{Im}x = \mp\rho\mathrm{Re}x\}$ for some $\rho > 0$, such that $x \longmapsto \mp\mathrm{Re}z(x)$ are strictly increasing functions along $\gamma_\pm(x)$. Denoting the path $z(\gamma_\pm(x))$ by $\Gamma_\pm(+\infty, z(x))$ and setting $w_{0,\pm} \equiv 1$, we inductively

define $w_{n,\pm}(z)$ by

$$\begin{aligned} w_{2n+1,\pm}(z) &= \int_{\Gamma_\pm(+\infty,z)} \exp(\pm\frac{2}{h}(\zeta - z))\frac{H'(\zeta)}{H(\zeta)}w_{2n,\pm}(\zeta)d\zeta, \\ w_{2n+2,\pm}(z) &= \int_{\Gamma_\pm(+\infty,z)} \frac{H'(\zeta)}{H(\zeta)}w_{2n+1,\pm}(\zeta)d\zeta, \quad n \geq 0. \end{aligned}$$

Noticing that

$$\frac{H'(x)}{H(x)} = \frac{mc^2}{2}\frac{V'(x)}{(V(x)-E)^2 - m^2c^4} = O(\langle x\rangle^{-\delta}), \ \ \delta > 1, \ \text{as } |x| \longrightarrow \infty,$$

one constructs well-defined complex WKB solutions $u^\pm_{r,l}$ corresponding to these base points, proceeding as in Section 3. Here, l and r stand for $left$ and $right$ and correspond respectively to $x \to -\infty$ and $x \to +\infty$. Up to a constant pre-factor, $u^\pm_{r,l}(x)$ are the previously defined Jost solutions:

Lemma 4.1. *Let $u^\pm_{r,l}(x)$ be the complex WKB solutions with phase and amplitude base point at infinity. Then*

$$u^\pm_r(x) \sim \exp(\pm\frac{1}{hc}(m^2c^4 - (V^+ - E)^2)^{1/2}x)\begin{pmatrix} \mp i\alpha^+ \\ 1/\alpha^+ \end{pmatrix}, \quad x \longrightarrow +\infty, \tag{4.3}$$

$$u^\pm_l(x) \sim \exp(\pm\frac{1}{hc}(m^2c^4 - (V^- - E)^2)^{1/2}x)\begin{pmatrix} \mp i\alpha^- \\ 1/\alpha^- \end{pmatrix}, \quad x \longrightarrow -\infty, \tag{4.4}$$

with

$$\alpha^\pm = \left(\frac{V^\pm - E - mc^2}{-V^\pm + E - mc^2}\right)^{1/4}.$$

Proof. We just check the asymptotic behavior of $u^\pm_{r,l}(x)$ at infinity. Since $H(z(x)) \longrightarrow \alpha^\pm$ as $x \longrightarrow \pm\infty$, using (3.12) and (3.13) we get by an elementary calculation

$$u^\pm_r(x) \sim \exp(\pm\frac{1}{hc}(m^2c^4 - (V^+ - E)^2)^{1/2}x)\begin{pmatrix} \mp i\alpha^+ & \pm i\alpha^+ \\ 1/\alpha^+ & 1/\alpha^+ \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ x \longrightarrow +\infty,$$

$$u^\pm_l(x) \sim \exp(\pm\frac{1}{hc}(m^2c^4 - (V^- - E)^2)^{1/2}x)\begin{pmatrix} \mp i\alpha^- & \pm i\alpha^- \\ 1/\alpha^- & 1/\alpha^- \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ x \longrightarrow -\infty.$$

This ends the proof of lemma. □

Let us now choose the determinations of $(m^2c^4 - (V^\pm - E)^2)^{1/2}$ and $\alpha^\pm$ according to the intervals on the E-axis. This fixes the choice of $u^\pm_{l,r}$ and we can construct $\omega^\pm_{\text{in}}$, $\omega^\pm_{\text{out}}$ satisfying (2.6) and (2.7).

1. For $E \in$ I, we choose $(m^2c^4 - (V^\pm - E)^2)^{1/2} \in i\mathbb{R}^+$, $\alpha^\pm \in e^{i\pi/4}\mathbb{R}^+$ and we denote

$$\omega^-_{\text{in}} := e^{i\pi/4}u^+_l, \quad \omega^+_{\text{in}} := -e^{i\pi/4}u^-_r, \quad \omega^+_{\text{out}} := e^{i\pi/4}u^+_r, \quad \omega^-_{\text{out}} := -e^{i\pi/4}u^-_l. \tag{4.5}$$

2. For $E \in$ II, we choose $(m^2c^4 - (V^- - E)^2)^{1/2} \in i\mathbb{R}^+$, $\alpha^- \in e^{i\pi/4}\mathbb{R}^+$, $(m^2c^4 - (V^+ - E)^2)^{1/2} \in \mathbb{R}^-$, $\alpha^+ \in \mathbb{R}^+$ and we denote

$$\omega^-_{\text{in}} := e^{i\pi/4}u^+_l, \quad \omega^-_{\text{out}} := -e^{i\pi/4}u^-_l, \quad \omega^+_d := u^+_r. \tag{4.6}$$

3. For $E \in$ III, we choose $(m^2c^4 - (V^\pm - E)^2)^{1/2} \in i\mathbb{R}^+$, $\alpha^\pm \in e^{\mp i\pi/4}\mathbb{R}^+$ and we denote

$$\omega^-_{\text{in}} := e^{i\pi/4}u^+_l, \quad \omega^+_{\text{in}} := -e^{-i\pi/4}u^+_r, \quad \omega^+_{\text{out}} := e^{-i\pi/4}u^-_r, \quad \omega^-_{\text{out}} := -e^{i\pi/4}u^-_l. \tag{4.7}$$

4. For $E \in \mathrm{IV}$, we choose $(m^2c^4 - (V^+ - E)^2)^{1/2} \in i\mathbb{R}^+$, $\alpha^+ \in e^{-i\pi/4}\mathbb{R}^+$, $(m^2c^4 - (V^- - E)^2)^{1/2} \in \mathbb{R}^-$, $\alpha^+ \in \mathbb{R}^+$ and we denote

$$\omega_{\text{in}}^+ := -e^{-i\pi/4}u_r^+, \quad \omega_{\text{out}}^+ := e^{-i\pi/4}u_r^-, \quad \omega_d^- := u_l^-. \tag{4.8}$$

5. For $E \in \mathrm{V}$, we choose $(m^2c^4 - (V^\pm - E)^2)^{1/2} \in i\mathbb{R}^+$, $\alpha^\pm \in e^{-i\pi/4}\mathbb{R}^+$ and we denote

$$\omega_{\text{in}}^- := e^{-i\pi/4}u_l^-, \quad \omega_{\text{in}}^+ := -e^{-i\pi/4}u_r^+, \quad \omega_{\text{out}}^+ := e^{-i\pi/4}u_r^-, \quad \omega_{\text{out}}^- := -e^{-i\pi/4}u_l^+. \tag{4.9}$$

Theorem 4.1. *For real E, (2.5) has solutions of the following form:*

1. For $E \in \mathrm{I}, \mathrm{III}$ or V, there are four Jost solutions $\omega_{\text{in}}^\pm$, $\omega_{\text{out}}^\pm$ which behave like

$$\omega_{\text{in}}^\pm \sim \exp\{\mp\frac{i}{hc}\Phi(E - V^\pm)x\}\begin{pmatrix} A(E - V^\pm) \\ \mp A(E - V^\pm)^{-1}\end{pmatrix} \quad \text{as } x \longrightarrow \pm\infty, \tag{4.10}$$

$$\omega_{\text{out}}^\pm \sim \exp\{\pm\frac{i}{hc}\Phi(E - V^\pm)x\}\begin{pmatrix} A(E - V^\pm) \\ \pm A(E - V^\pm)^{-1}\end{pmatrix} \quad \text{as } x \longrightarrow \pm\infty, \tag{4.11}$$

with $\Phi(E) = \mathrm{sgn}(E)\sqrt{E^2 - m^2c^4}$, $A(E) = \sqrt[4]{\frac{E+mc^2}{E-mc^2}}$ and $\mathrm{sgn}(E) = \frac{E}{|E|}$ for $E \notin [-mc^2, mc^2]$.

2. For $E \in \mathrm{II}$ (resp. $E \in \mathrm{IV}$), there are two Jost solutions ω_{in}^-, ω_{out}^- (resp. ω_{in}^+, ω_{out}^+) which behave as in (4.10), (4.11) and a decreasing solution ω_d^+ (resp. ω_d^-). The solutions $\omega_d^\pm$ behave exactly like

$$\omega_d^\pm \sim \exp\{\mp\frac{1}{hc}\sqrt{m^2c^4 - (V^\pm - E)^2}x\}\begin{pmatrix} \mp i\sqrt[4]{\frac{mc^2+E-V^\pm}{mc^2-E+V^\pm}} \\ \sqrt[4]{\frac{mc^2-E+V^\pm}{mc^2+E-V^\pm}}\end{pmatrix} \quad \text{as } x \longrightarrow \pm\infty. \tag{4.12}$$

For $E \in \mathrm{I}, \mathrm{III}$ or V, according to the relation (2.13), it is sufficient to calculate the two terms $r(E,h)$, $t(E,h)$ in $\mathbb{T}$ to obtain the matrix $\mathbb{S}$. The definition of the Wronskian (see Definition 3.4) leads to:

$$t(E,h) = \frac{\mathcal{W}(\omega_{\text{in}}^-, \omega_{\text{in}}^+)}{\mathcal{W}(\omega_{\text{out}}^+, \omega_{\text{in}}^+)}, \tag{4.13}$$

$$r(E,h) = \frac{\mathcal{W}(\omega_{\text{out}}^-, \omega_{\text{in}}^+)}{\mathcal{W}(\omega_{\text{out}}^+, \omega_{\text{in}}^+)}. \tag{4.14}$$

5. The Klein paradox case

We suppose that V satisfies assumption (A), the energy $E \in \mathrm{III}$ and $m > 0$ (see Fig. 2). In this section we will work in two unbounded, simply-connected domains $\Omega^-(E)$, $\Omega^+(E)$, where $\mathrm{Re}\,(V(x) + mc^2) < E$, $\mathrm{Re}\,(V(x) - mc^2) > E$ respectively and which coincide with $\mathcal{S}$ for $|\mathrm{Re}\, x|$ sufficiently large. Using Theorem 4.1, Proposition 3.1 and (4.7) there are two Jost solutions in $\Omega^\pm(E)$:

$$\omega_{\text{in}}^\pm = \exp\{\frac{1}{h}z(x, \pm\infty)\}\begin{pmatrix} \widetilde{H}(z(x)) \\ \mp\widetilde{H}(z(x))^{-1}\end{pmatrix}(1 + h\phi(h)) \tag{5.1}$$

$$\omega_{\text{out}}^\pm = \exp\{\frac{-1}{h}z(x, \pm\infty)\}\begin{pmatrix} \widetilde{H}(z(x)) \\ \pm\widetilde{H}(z(x))^{-1}\end{pmatrix}(1 + h\phi(h)). \tag{5.2}$$

The function $z(x, \pm\infty)$ is defined by (4.1) and

$$\widetilde{H}(z(x)) = \left(\frac{E - V(x) + mc^2}{E - V(x) - mc^2}\right)^{\frac{1}{4}}. \tag{5.3}$$

On $\Omega^{\pm}(E)\cap\mathbb{R}$, we have:

$$z(x,\pm\infty) = i\int_{\pm\infty}^{x}\sqrt{\frac{(E-V(t))^2-m^2c^4}{c^2}}-\sqrt{\frac{(E-V^{\pm})^2-m^2c^4}{c^2}}\,dt \quad (5.4)$$
$$+i\sqrt{\frac{(E-V^{\pm})^2-m^2c^4}{c^2}}\,x$$

$$\widetilde{H}(z(x)) = \sqrt[4]{\frac{E-V(x)+mc^2}{E-V(x)-mc^2}}. \quad (5.5)$$

We suppose that there are only two real turning points $t_1(E) < t_2(E)$ and that they are simple. Notice that $t_1(E)$ is a zero of $E-V(t)-mc^2$ and $t_2(E)$ is a zero of $E-V(t)+mc^2$. In that case the Stokes lines are as shown in Fig. 4. In order to obtain $\mathbb{S}$, we compute the Wronskians given in (4.13), (4.14) and then the coefficients $t(E,h)$, $r(E,h)$.

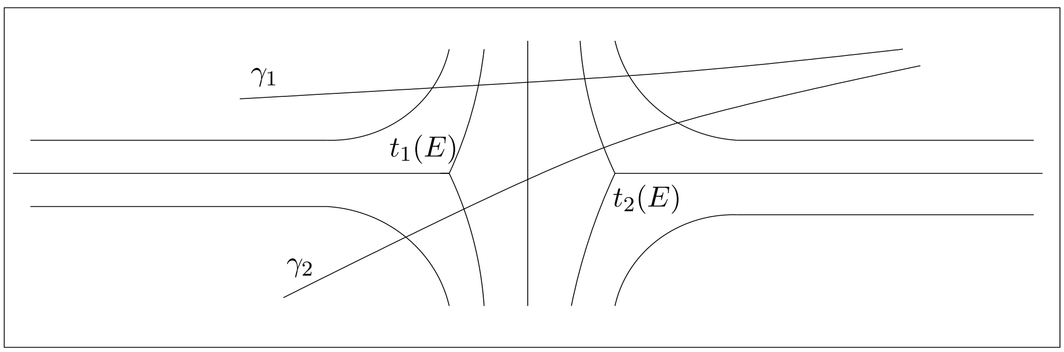

Fig. 4. The turning points and the paths γ_j

Computation of $\mathcal{W}(\omega^+_{\text{out}},\omega^+_{\text{in}})$*:* Since the two solutions $\omega^+_{\text{out}},\omega^+_{\text{in}}$ are defined in $\Omega^+(E)$, we can compute this Wronskian in $\Omega^+(E)$ and from Lemma 3.2 we obtain

$$\mathcal{W}(\omega^+_{\text{out}},\omega^+_{\text{in}}) = -2. \quad (5.6)$$

Computation of $\mathcal{W}(\omega^-_{\text{in}},\omega^+_{\text{in}})$*:* The two solutions $\omega^-_{\text{in}},\omega^+_{\text{in}}$ are defined in $\Omega^-(E),\Omega^+(E)$ respectively. Since the Wronkians $\mathcal{W}(\omega^-_{\text{in}},\omega^+_{\text{in}})(x)$ are independent on x (see Remark 3.1) we compute this Wronskian in $\Omega^-(E)$ for example. For that we extend ω^+_{in} , which is defined in $\Omega^+(E)$, into $\Omega^-(E)$. We will extend the square root in ω^+_{in} which is defined in $\Omega^+(E)$, into $\mathbb{C}\setminus\{-\text{Im }(z)>0,\ \text{Re }(z)=t_1(E)\}\cup\{-\text{Im }(z)>0,\ \text{Re }(z)=t_2(E)\}$. Thanks to the structure of the Stokes lines between $t_1(E)$ and $t_2(E)$, we can find a path γ_1 from $+\infty(1+i\delta_1)$ to $-\infty(1-i\delta_1)$ (for $\delta_1>0$) transverse to the Stokes lines along which we can extend ω^+_{in}. We remark that between $t_1(E)$ and $t_2(E)$ on the real axis we have $(E-V(t))^2-m^2c^4<0$. The extension of $t\in]t_2(E),+\infty[\longrightarrow\sqrt{\frac{(E-V(t))^2-m^2c^4}{c^2}}$ coincide with $i\sqrt{\frac{m^2c^4-(E-V(t))^2}{c^2}}$ on $]t_1(E),t_2(E)[$ and with $-\sqrt{\frac{(E-V(t))^2-m^2c^4}{c^2}}$ on $]-\infty,t_1(E)[$. On the other hand, the extension of $\widetilde{H}(z(x))$ stay in $\mathbb{R}^+$ on $]-\infty,t_1(E)[$. If we denote by $\omega^{+,1}_{\text{in}}$ the extension of ω^+_{in} along γ_1, we have:

$$\omega^{+,1}_{\text{in}} = \exp\{\frac{1}{h}(-z(x,t_1(E))+z(t_2(E),+\infty)+S(E))\}\begin{pmatrix}\widetilde{H}(z(x))\\ -\widetilde{H}(z(x))^{-1}\end{pmatrix}(1+h\phi(h)),$$

with $z(t_2(E),+\infty)$ defined in (5.4) and

$$z(x,t_1(E)) = i\int_{t_1(E)}^{x}\left(\frac{(E-V(t))^2-m^2c^4}{c^2}\right)^{\frac{1}{2}}dt, \tag{5.7}$$

$$S(E) = \int_{t_1(E)}^{t_2(E)}\sqrt{\frac{m^2c^4-(E-V(t))^2}{c^2}}dt. \tag{5.8}$$

Here, $\left(\frac{(E-V(t))^2-m^2c^4}{c^2}\right)^{\frac{1}{2}} \in \mathbb{R}^+$ for $t\in]-\infty,t_1(E)[$.

Then,

$$\mathcal{W}(\omega_{\text{in}}^-,\omega_{\text{in}}^+) = -2(1+h\phi(h))\exp\{\frac{1}{h}(z(t_1(E),-\infty)+z(t_2(E),+\infty)+S(E))\},$$

where $z(t_2(E),+\infty)$, $z(t_1(E),-\infty)$ are defined in (5.4).

Computation of $\mathcal{W}(\omega_{\text{out}}^-,\omega_{\text{in}}^+)$*:* This wronskian is also between two solutions which are defined in different domains, then we extend one of these solutions into the domain of the other solution. For example we extend ω_{in}^+, which is defined in $\Omega^+(E)$, into $\Omega^-(E)$ which is a subset of $\mathbb{C}\setminus\{\text{Im }(z)>0,\ \text{Re }(z)=t_1(E)\}\cup\{-\text{Im }(z)>0,\ \text{Re }(z)=t_2(E)\}$. Here, we can also find a path γ_2 from $+\infty(1+i\delta_2)$ to $-\infty(1+i\delta_2)$ for $\delta_2>0$ transverse to the Stokes lines along which we can extend ω_{in}^+ into $\Omega^-(E)$. If we denote by $\omega_{\text{in}}^{+,2}$ the extension of ω_{in}^+ along γ_2, we have:

$$\omega_{\text{in}}^{+,2} = \exp\{\frac{1}{h}(z(x,t_1(E))+z(t_2(E),+\infty)+S(E))\}\begin{pmatrix} i\widetilde{H}(z(x)) \\ i\widetilde{H}(z(x))^{-1}\end{pmatrix}(1+h\phi(h)).$$

Here $\widetilde{H}(z(x))\in\mathbb{R}^+$ on $]-\infty,t_1(E)[$ and $z(x,t_1(E))$, $S(E)$ are defined in (5.7), (5.8).

The computation of $\mathcal{W}(\omega_{\text{out}}^-,\omega_{\text{in}}^+)$ yields:

$$\mathcal{W}(\omega_{\text{out}}^-,\omega_{\text{in}}^+) = 2i(1+h\phi(h))\exp\{\frac{1}{h}(-z(t_1(E),-\infty)+z(t_2(E),+\infty)+S(E))\}.$$

Then, we obtain (see (4.13) and (4.14)):

$$\begin{aligned} t(E,h) &= (1+h\phi(h))\exp\{\frac{1}{h}(z(t_1(E),-\infty)+z(t_2(E),+\infty)+S(E))\},\\ r(E,h) &= -i(1+h\phi(h))\exp\{\frac{1}{h}(-z(t_1(E),-\infty)+z(t_2(E),+\infty)+S(E))\}.\end{aligned}$$

Since $\phi(h)$ is a classical analytic symbols of non-negative order and using (2.13) we have:

$$\begin{aligned} s_{11} &= \frac{1}{\overline{t}(E,h)} = (1+h\phi_1(h))\exp\{\frac{1}{h}(z(t_1(E),-\infty)+z(t_2(E),+\infty)-S(E))\},\\ s_{21} &= \frac{\overline{r}(E,h)}{\overline{t}(E,h)} = (i+h\phi_2(h))\exp\{\frac{2}{h}(z(t_1(E),-\infty)\},\\ s_{12} &= \frac{-r(E,h)}{\overline{t}(E,h)} = (i+h\phi_3(h))\exp\{\frac{2}{h}(z(t_2(E),+\infty)\}.\end{aligned}$$

The functions $\phi_1(h)$, $\phi_2(h)$, $\phi_3(h)$ are classical analytic symbols of non-negative order. This ends the proof of Theorem 2.1.

6. Total transmission

We suppose that V satisfies assumption (A), the energy $E \in \mathrm{I}$ or $E \in \mathrm{V}$ and $m \geq 0$ (see Fig. 5, Fig. 6).

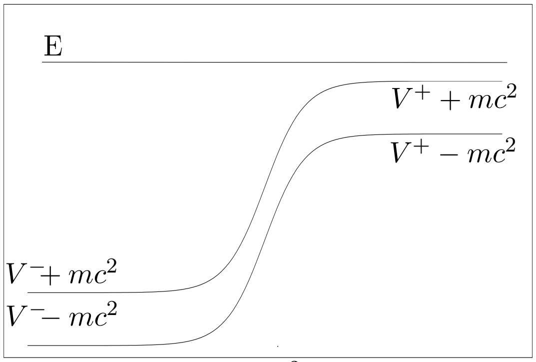

Fig. 5. $V \pm mc^2$ and $E \in$I

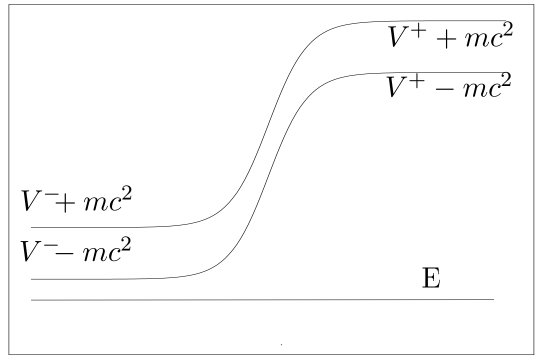

Fig. 6. $V \pm mc^2$ and $E \in$V

We suppose that there exists no real turning point. In that case the Stokes lines are horizontal lines near the real axis. We will only work for $E \in \mathrm{I}$. The case where $E \in \mathrm{V}$ can be treated similarly.

In this section we work in $\Omega^-(E)$ defined in the previous section. Now this set is a neighborhood of the real axis. Using Theorem 4.1, Proposition 3.1 and (4.5) there are four Jost solutions:

$$\begin{aligned} \omega_{\text{in}}^{\pm} &= \exp\{\frac{\mp 1}{h} z(x, \pm\infty)\} \begin{pmatrix} \widetilde{H}(z(x)) \\ \mp\widetilde{H}(z(x))^{-1} \end{pmatrix} (1 + h\phi(h)) \\ \omega_{\text{out}}^{\pm} &= \exp\{\frac{\pm 1}{h} z(x, \pm\infty)\} \begin{pmatrix} \widetilde{H}(z(x)) \\ \pm\widetilde{H}(z(x))^{-1} \end{pmatrix} (1 + h\phi(h)). \end{aligned} \tag{6.1}$$

The functions $z(x, \pm\infty)$ and $\widetilde{H}(z(x))$ are defined in (4.1) and (5.3) and coincide with (5.4), (5.5) on the real axis. Here, the setting is different from the previous section. The solutions $\omega_{\text{in}}^{\pm}$ and $\omega_{\text{out}}^{\pm}$ are defined in the same domain $\Omega^-(E)$ and there are no problem to extend the different square roots.

As in Section 4, it is sufficient to calculate the two terms $r(E, h)$, $t(E, h)$ (see (4.13), (4.14)) to obtain the matrix $\mathbb{S}$.

Computation of $\mathcal{W}(\omega_{\text{in}}^-, \omega_{\text{in}}^+)$, $\mathcal{W}(\omega_{\text{out}}^+, \omega_{\text{in}}^+)$: Since the function $\widetilde{H}$ in (6.1) is the same for ω_{in}^- and ω_{in}^+, we have:

$$\begin{aligned} \mathcal{W}(\omega_{\text{in}}^-, \omega_{\text{in}}^+)(x) &= -2(1 + h\phi(h)) \exp\{\frac{1}{h}(z(x, +\infty) - z(x, -\infty))\} \\ &= -2(1 + h\phi(h)) \exp\{\frac{1}{h}(z(0, +\infty) - z(0, -\infty))\}. \end{aligned} \tag{6.2}$$

Moreover, as in (5.6)

$$\mathcal{W}(\omega_{\text{out}}^+, \omega_{\text{in}}^+)(x) = -2.$$

Then, we obtain (see (4.13))

$$t(E, h) = (1 + h\phi(h)) \exp\{\frac{1}{h}(z(0, +\infty) - z(0, -\infty))\}, \tag{6.3}$$

with

$$z(0,\pm\infty) = i\int_{\pm\infty}^{0} \sqrt{\frac{(E-V(t))^2-m^2c^4}{c^2}} - \sqrt{\frac{(E-V^{\pm})^2-m^2c^4}{c^2}}\, dt. \tag{6.4}$$

Since $\phi(h)$ is a classical analytic symbol of non-negative order and using (2.13) we have:

$$s_{11} \;=\; \frac{1}{\overline{t}(E,h)} = (1+h\tilde{\phi}(h))\exp\{\frac{1}{h}(z(0,+\infty)-z(0,-\infty))\}.$$

Computation of $\mathcal{W}(\omega^-_{\text{out}},\omega^+_{\text{in}})$*:* As in (6.2),

$$\mathcal{W}(\omega^-_{\text{out}},\omega^+_{\text{in}})(x) = O(h)\exp\{-\frac{1}{h}(z(x,+\infty)+z(x,-\infty))\}.$$

Using that the square root in $z(x,+\infty)$ and $z(x,-\infty)$ have the same determination, we have

$$\mathcal{W}(\omega^-_{\text{out}},\omega^+_{\text{in}})(x) = O(h)\exp\{-\frac{1}{h}(z(0,+\infty)+z(0,-\infty))\}\exp\{-\frac{2}{h}(z(x,0))\},$$

where $z(0,\pm\infty)\in i\mathbb{R}$ is defined in (6.4) and $z(x,0) = i\int_0^x \sqrt{\frac{(E-V(t))^2-m^2c^4}{c^2}}dt$. Since the Wronskians are independent on x, we estimate the term $z(x,0)$ for $x=-iy$, $0<y\ll 1$. Here, we have $z(x,0) = z(-iy,0) = -iy(i\sqrt{\frac{(E-V(0))^2-m^2c^4}{c^2}}) + O(y^2) = Cy + O(y^2)$ for $C>0$. Thereafter, $\mathcal{W}(\omega^-_{\text{out}},\omega^+_{\text{in}}) = O(e^{-C/h})$ for an other $C>0$ and then

$$r(E,h) = O(e^{-C/h}). \tag{6.5}$$

Consequently, using (2.13), we have, for a positive constant C,

$$s_{21} = \frac{\overline{r}(E,h)}{\overline{t}(E,h)} = O(e^{-C/h}),$$

$$s_{12} = \frac{-r(E,h)}{\overline{t}(E,h)} = O(e^{-C/h}).$$

This ends the proof of Theorem 2.4.

7. Total reflection

We suppose here that V satisfies assumption (A), the energy $E\in$ II or IV and $m>0$ (see Fig. 7 or Fig. 8).

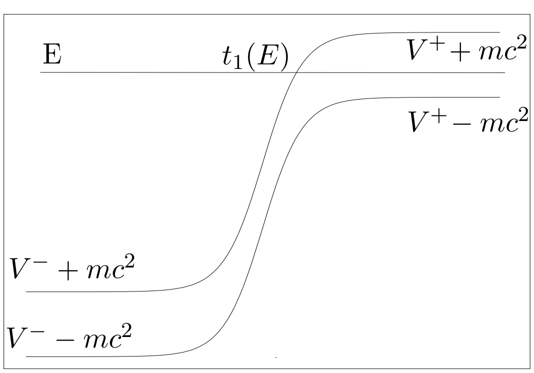

Fig. 7. $E\in$II

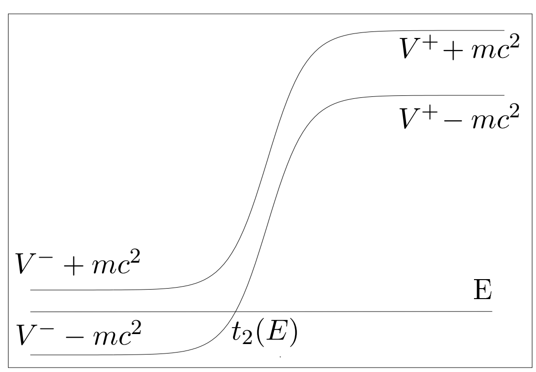

Fig. 8. $E\in$IV

As in the Section 5, we will work in two unbounded, simply-connected domains $\Omega^-(E)$, $\Omega^+(E)$, where Re $(V(x) - E + mc^2) < E$, Re $(V(x) - mc^2) > 0$ respectively and which coincide with $\mathcal{S}$ for $|\mathrm{Re}\ x|$ sufficiently large. We will only work for $E \in$ II. The case $E \in$ IV can be treated similarly. Using Theorem 4.1, Proposition 3.1 and (4.6) there are two Jost solutions in $\Omega^-(E)$:

$$\begin{aligned}\omega_{\mathrm{in}}^- &= \exp\{\frac{1}{h} z(x,-\infty)\} \begin{pmatrix} \widetilde{H}(z(x)) \\ +\widetilde{H}(z(x))^{-1} \end{pmatrix} (1 + h\phi(h)) \\ \omega_{\mathrm{out}}^- &= \exp\{\frac{-1}{h} z(x,-\infty)\} \begin{pmatrix} \widetilde{H}(z(x)) \\ -\widetilde{H}(z(x))^{-1} \end{pmatrix} (1 + h\phi(h))\end{aligned}$$

with $\phi(h)$ a classical analytic symbol of non-negative order. The functions $z(x,-\infty)$, $\widetilde{H}(z(x))$ are defined in (4.1), (5.3) and coincide with (5.4), (5.5) on the real axis. From Lemma 4.1, there exist an exponentially decreasing Jost solution and an exponentially increasing one. As explained before Theorem 2.3, we exclude the increasing solutions which does not represent a physical state. We limit ourself to the one-dimensional space generated by the decreasing solution ω_d^+ which satisfies in $\Omega^+(E)$:

$$\omega_d^+ = \exp\{\frac{-1}{h} z(x,+\infty)\} \begin{pmatrix} -iH(z(x)) \\ H(z(x))^{-1} \end{pmatrix} (1 + h\phi(h)), \tag{7.1}$$

from Theorem 4.1, Proposition 3.1 and (4.6). The functions $z(x,+\infty)$, $H(z(x))$ are defined in (4.1), (3.5) respectively and coincide, on the real axis, with

$$\begin{aligned} z(x,+\infty) &= \frac{1}{c}\int_{+\infty}^{x} \sqrt{m^2c^4 - (E - V(t))^2} - \sqrt{m^2c^4 - (E - V^+)^2}\, dt \\ &\quad + \frac{1}{c}\sqrt{m^2c^4 - (E - V^+)^2}\ x \end{aligned} \tag{7.2}$$

$$H(z(x)) = \sqrt[4]{\frac{mc^2 + E - V(x)}{mc^2 - E + V(x)}}. \tag{7.3}$$

We suppose that there is only one real turning point $t_1(E)$ and that it is simple. In that case the Stokes lines are as shown in the fig. Fig. 9.

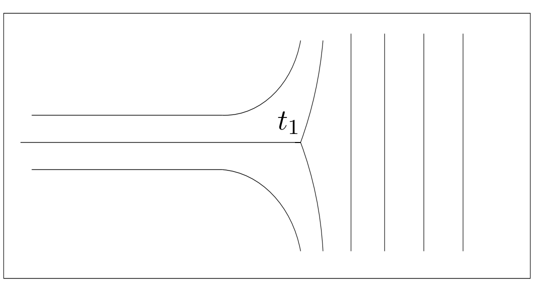

Fig. 9. Stokes lines for $E \in$II

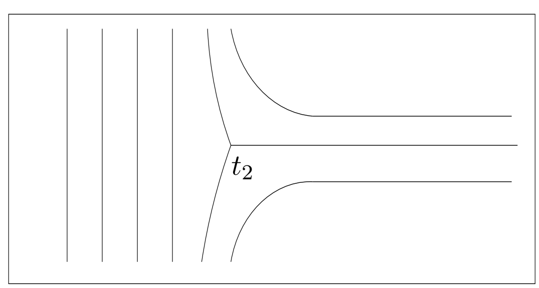

Fig. 10. Stokes lines for $E \in$IV

According to the definition of the Wronskian, we have

$$\alpha_{\mathrm{out}}^- = \frac{\mathcal{W}(\omega_{\mathrm{in}}^-, \omega_d^+)}{\mathcal{W}(\omega_d^+, \omega_{\mathrm{out}}^-)}, \qquad \beta_d^+ = \frac{\mathcal{W}(\omega_{\mathrm{in}}^-, \omega_{\mathrm{out}}^-)}{\mathcal{W}(\omega_d^+, \omega_{\mathrm{out}}^-)}.$$

Computation of $\mathcal{W}(\omega_{\rm in}^-,\omega_d^+)$*:* In order to calculate this Wronskian we need to extend one of the solutions $\omega_{\rm in}^-$, ω_d^+ from its domain to the domain of the other solution, for example, we extend ω_d^+ from $\Omega^+(E)$ to $\Omega^-(E)$. For that, we extend the square root in ω_d^+ to $\mathbb{C}\backslash\{-{\rm Im}\ (z)>0,\ {\rm Re}\ (z)=t_1\}$. Thanks to the structure of the Stokes lines , we can find a path $\tilde{\gamma}_1$ from $+\infty(1-i\tilde{\delta}_1)$ to $-\infty(1-i\tilde{\delta}_1)$ (for $\tilde{\delta}_1>0$) transverse to the Stokes lines along which we can extend ω_d^-. The extension of $t\in]t_1(E),+\infty[\longrightarrow\sqrt{m^2c^4-(E-V(t))^2}$ coincides with $i\sqrt{(E-V(t))^2-m^2c^4}$ on $]-\infty,t_1(E)[$. On the other hand, on $]-\infty,t_1(E)[$, $H(z(x))$ takes its values in $e^{-i\pi/4}\mathbb{R}^+$. If we denote by $\omega_d^{+,1}$ the extension of ω_d^+ along $\tilde{\gamma}_1$, we have

$$\omega_d^{+,1}=\exp\{\frac{1}{h}(-z(x,t_1(E))-z(t_1(E),+\infty))\}\begin{pmatrix}-e^{i\pi/4}\widetilde{H}(z(x))\\ e^{i\pi/4}\widetilde{H}(z(x))^{-1}\end{pmatrix}(1+h\phi(h)),$$

with $z(t_1(E),+\infty)$, $\widetilde{H}(z(x))$ defined in (7.2), (5.3) and

$$z(x,t_1(E)) = \frac{i}{c}\int_{t_1(E)}^{x}((E-V(t))^2-m^2c^4)^{\frac12}dt. \tag{7.4}$$

On $]-\infty,t_1(E)[$ the functions $((E-V(t))^2-m^2c^4)^{\frac12}$ and $\widetilde{H}(z(x))$ are in $\mathbb{R}^+$. Then, we have

$$\mathcal{W}(\omega_{\rm in}^-,\omega_d^+)=2e^{i\pi/4}(1+h\phi(h))\exp\{\frac{1}{h}(z(t_1(E),-\infty)-z(t_1(E),+\infty))\},$$

where, $z(t_1,+\infty)$, $z(t_1(E),-\infty)$ are defined respectively in (7.2) and (5.4).

Computation of $\mathcal{W}(\omega_d^+,\omega_{\rm out}^-)$*:* As in the previous paragraph we extend ω_d^+ and the square roots written there from $\Omega^+(E)$ to $\Omega^-(E)\subset\mathbb{C}\setminus\{{\rm Im}\ (z)>0,\ {\rm Re}\ (z)=t_1\}$. We can also find a path $\tilde{\gamma}_2$ from $+\infty(1-i\tilde{\delta}_2)$ to $-\infty(1+i\tilde{\delta}_2)$ (for $\tilde{\delta}_2>0$) transverse to the Stokes lines along which we can extend ω_d^+. If we denote by $\omega_d^{+,2}$ the extension of ω_d^+ along $\tilde{\gamma}_2$, we have

$$\omega_d^{+,2}=\exp\{\frac{1}{h}(+z(x,t_1(E))-z(t_1(E),+\infty))\}\begin{pmatrix}e^{-i\pi/4}\widetilde{H}(z(x))\\ e^{-i\pi/4}\widetilde{H}(z(x))^{-1}\end{pmatrix}(1+h\phi(h)),$$

with $z(x,t_1(E))$, $z(t_1(E),+\infty)$ and $\widetilde{H}(z(x))$ defined respectively in (7.4), (7.2) and (5.3). On $]-\infty,t_1(E)[$ the quantities $((E-V(t))^2-m^2c^4)^{\frac12}$ and $\widetilde{H}(z(x))$ are in $\mathbb{R}^+$. Then, we have

$$\mathcal{W}(\omega_d^+,\omega_{\rm out}^-)=-2e^{-i\pi/4}(1+h\phi(h))\exp\{\frac{1}{h}(-z(t_1(E),-\infty)-z(t_1,+\infty))\},$$

where, $z(t_1,+\infty)$, $z(t_1(E),-\infty)$ are defined respectively in (7.2) and (5.4).

Computation of $\mathcal{W}(\omega_{\rm in}^-,\omega_{\rm out}^-)$*:* Since the two solutions $\omega_{\rm in}^-,\omega_{\rm out}^-$ are defined in $\Omega^-(E)$, we compute the Wronskian between these solutions as in (5.6) and obtain

$$\mathcal{W}(\omega_{\rm in}^-,\omega_{\rm out}^-)=-2.$$

Then, we have

$$\begin{aligned}\alpha_{\rm out}^- &= \frac{\mathcal{W}(\omega_{\rm in}^-,\omega_d^+)}{\mathcal{W}(\omega_d^+,\omega_{\rm out}^-)}=-i(1+h\phi(h))\exp\{\frac{2}{h}z(t_1(E),-\infty)\}\\ \beta_d^+ &= \frac{\mathcal{W}(\omega_{\rm in}^-,\omega_{\rm out}^-)}{\mathcal{W}(\omega_d^+,\omega_{\rm out}^-)}=e^{i\pi/4}(1+h\phi(h))\exp\{\frac{1}{h}(z(t_1(E),-\infty)+z(t_1,+\infty))\},\end{aligned}$$

with

$$\begin{aligned}
z(t_1(E),-\infty) &= \frac{i}{c}\int_{-\infty}^{t_1(E)} \sqrt{(E-V(t))^2-m^2c^4}-\sqrt{(E-V^-)^2-m^2c^4}\,dt \\
&+ \frac{i}{c}\sqrt{(E-V^-)^2-m^2c^4}\;t_1(E) \\
z(t_1(E),+\infty) &= \frac{1}{c}\int_{+\infty}^{t_1(E)} \sqrt{m^2c^4-(E-V(t))^2}-\sqrt{m^2c^4-(E-V^+)^2}\,dt \\
&+ \frac{1}{c}\sqrt{m^2c^4-(E-V^+)^2}\;t_1(E).
\end{aligned}$$

This ends the proof of Theorem 2.3.

8. Zero mass case

We suppose that $m=0$, $E\in]V^-,V^+[$ and V satisfies assumption (A),(see Fig. 11).

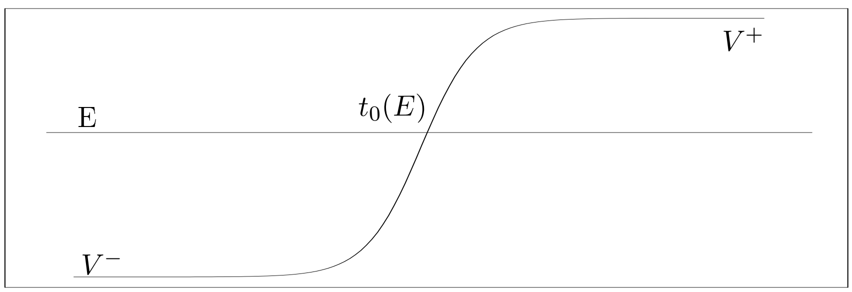

Fig. 11. The double turning point t_0

In this section we will work in two unbounded, simply-connected domains $\Omega_0^-(E)$, $\Omega_0^+(E)$, where Re $V(x)<E$, Re $V(x)>E$ respectively and which coincide with $\mathcal{S}$ for $|\mathrm{Re}\,x|$ sufficiently large. Repeating the constructions of the solutions (see Sections 3, 4) and using Proposition 3.1 for $m=0$, there are two Jost solutions in $\Omega_0^\pm(E)$:

$$\begin{aligned}
\omega_{\mathrm{in}}^\pm &= \exp\{\frac{1}{h}z_0(x,\pm\infty)\}\begin{pmatrix}1\\ \mp 1\end{pmatrix}(1+h\phi(h)) \\
&\sim \exp\{\frac{\pm i}{hc}(V^\pm-E)x\}\begin{pmatrix}1\\ \mp 1\end{pmatrix}\quad \text{as } x\longrightarrow\pm\infty, \\
\omega_{\mathrm{out}}^\pm &= \exp\{\frac{-1}{h}z_0(x,\pm\infty)\}\begin{pmatrix}1\\ \pm 1\end{pmatrix}(1+h\phi(h)) \\
&\sim \exp\{\frac{\mp i}{hc}(V^\pm-E)\,x\}\begin{pmatrix}1\\ \pm 1\end{pmatrix}\quad \text{as } x\longrightarrow\pm\infty,
\end{aligned}\tag{8.1}$$

with $\phi(h)$ a classical analytic symbol of non-negative order. The functions $z_0(x,\pm\infty)$ are defined by

$$z_0(x,\pm\infty) = \pm\frac{i}{c}\int_{\pm\infty}^{x}(V(t)-V^\pm)dt \pm \frac{i}{c}(V^\pm-E)x.$$

We suppose that there is only a simple zero of $V(x)-E$ (see Fig. 11). Recall that the turning points and the Stokes lines are also defined for $m=0$ (see Definition 3.1, Definition

3.2). In our setting, there exists only a double turning point $t_0(E)$ and the Stokes lines are described in Fig. 12.

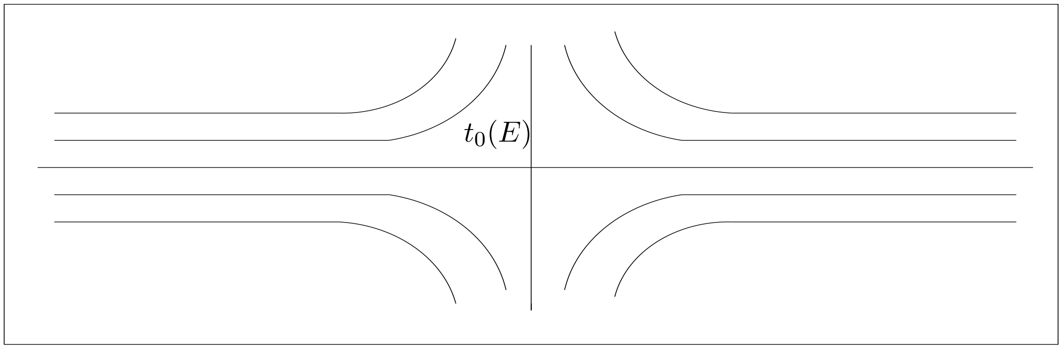

Fig. 12. The Stokes lines and the double turning point

In this section we use the definition of the scattering and transfer matrix of Section 2 with the incoming and outgoing Jost solutions defined above. As in Section 4, it is sufficient to calculate the two terms $r(E,h)$, $t(E,h)$ (see (4.13), (4.14)) to obtain the matrix $\mathbb{S}$.

Computation of $\mathcal{W}(\omega_{\text{out}}^+, \omega_{\text{in}}^+)$*:* Since the two solutions ω_{out}^+, ω_{in}^+ are defined in Ω_0^+, we compute this Wronskian in Ω_0^+ as (5.6) and we obtain

$$\mathcal{W}(\omega_{\text{out}}^+, \omega_{\text{in}}^+) = -2.$$

Computation of $\mathcal{W}(\omega_{\text{in}}^-, \omega_{\text{in}}^+)$*:* The two solutions $\omega_{\text{in}}^-, \omega_{\text{in}}^+$ are defined in $\Omega_0^-(E), \Omega_0^+(E)$ respectively. Since the Wronkian $\mathcal{W}(\omega_{\text{in}}^-, \omega_{\text{in}}^+)(x)$ is independent on x (see Remark 3.1), we compute it in $\Omega_0^-(E)$ for example. For that, we extend ω_{in}^+ from $\Omega_0^+(E)$ to $\Omega_0^-(E)$. Using (8.1) we get

$$\begin{aligned}
\mathcal{W}(\omega_{\text{in}}^-, \omega_{\text{in}}^+) &= -2(1+h\phi(h)) \exp\left(\frac{1}{h}\left(z(x,-\infty) + z(x,+\infty)\right)\right), \\
&= -2(1+h\phi(h)) \exp\left(\frac{1}{h}\left(z(t_0(E),-\infty) + z(t_0(E),+\infty)\right)\right).
\end{aligned}$$

Then, we have

$$\begin{aligned}
t(E,h) &= \frac{\mathcal{W}(\omega_{\text{in}}^-, \omega_{\text{in}}^+)}{\mathcal{W}(\omega_{\text{out}}^+, \omega_{\text{in}}^+)}, \\
&= (1+h\phi(h)) \exp\left(\frac{1}{h}\left(z(t_0(E),-\infty) + z(t_0(E),+\infty)\right)\right).
\end{aligned} \tag{8.2}$$

From (2.13), it follows

$$s_{11} = \frac{1}{\overline{t}(E,h)} = (1+h\phi_0(h)) \exp\{\frac{i}{h} T_0(E)\},$$

with

$$T_0(E) = \frac{1}{c}\left(\int_{+\infty}^{t_0(E)} (V(t)-V^+)dt - \int_{-\infty}^{t_0(E)} (V(t)-V^-)dt + t_0(E)(V^+ - V^-)\right).$$

From the form of the Stokes lines (see Fig. 12) there exist no path transverse to the Stokes lines along which we calculate the Wronkian $\mathcal{W}(\omega_{\text{out}}^-, \omega_{\text{in}}^+)$. More precisely, the WKB method

does not give the asymptotic behavior of the Wronkian $\mathcal{W}(\omega^-_{\text{out}}, \omega^+_{\text{in}})$ as well as the term

$$r(E,h) = \frac{\mathcal{W}(\omega^-_{\text{out}}, \omega^+_{\text{in}})}{\mathcal{W}(\omega^+_{\text{out}}, \omega^+_{\text{in}})}.$$

Nevertheless, (8.2) together with the relation (2.12), implies:

$$r(E,h) = O(h). \tag{8.3}$$

Consequently, using (2.13), we have

$$s_{21} = \frac{\overline{r}(E,h)}{\overline{t}(E,h)} = O(h),$$

$$s_{12} = \frac{-r(E,h)}{\overline{t}(E,h)} = O(h).$$

This end the proof of Theorem 2.2.

Appendix A. Spectrum of the Dirac operator

Proposition A.1. *Suppose that $V(x)$ is a L^∞ application with values in the space of Hermitian $2-$matrix. Moreover we assume that*

$$\|V(x) - V^\pm I_2\| \to 0, \quad \textit{as } x \to \pm\infty.$$

Then the operator $H = H_0 + V$ is a selfadjoint operator on $D(H_0)$ and

$$\sigma_{ess}(H) \;=\;]-\infty, -mc^2 + V^+] \cup [mc^2 + V^-, +\infty[. \tag{A.1}$$

Proof. In order to prove this proposition, we first calculate the essential spectrum of $H_0 + W$, where W is a L^∞ potential with $W(x) = V^\pm I_2$ for $\pm x > R > 0$. From Lemma 5.1 of [19], we know the following inclusion:

$$\sigma_{ess}(H_0 + W) \subset]-\infty, -mc^2 + V^+] \cup [mc^2 + V^-, +\infty[. \tag{A.2}$$

Let us now prove the second inclusion. We denote

$$I^+ :=]-\infty, -mc^2 + V^+], \quad \text{and} \quad I^- := [mc^2 + V^-, +\infty[.$$

For $E \in I^\pm$, we consider the sequence

$$f_n^\pm = \exp\{\mp\frac{i}{hc}\Phi(E - V^\pm)x\} \begin{pmatrix} A(E - V^\pm) \\ \mp A(E - V^\pm)^{-1} \end{pmatrix} \chi(\pm x/n) \;\text{ for }\; n \in \mathbb{N},$$

with $\Phi(E) = sgn(E)\sqrt{E^2 - m^2c^4}$, $A(E) = \sqrt[4]{\frac{E+mc^2}{E-mc^2}}$ and $sgn(E) = \frac{E}{|E|}$ for $E \notin [-mc^2, mc^2]$. The function $\chi \in C_0^\infty(\mathbb{R})$ is such that $\chi(x) = 1$ if $2 < x < 3$ and $\chi(x) = 0$ if $x < 1$ and $x > 4$.

The normed sequence $(\frac{f_n^\pm}{\|f_n^\pm\|})_{n\in\mathbb{N}}$ has no convergent subsequence and satisfies

$$(H_0 + W - E)\frac{f_n^\pm}{\|f_n^\pm\|} \longrightarrow 0, \quad \text{as } n \longrightarrow +\infty.$$

From the Weyl criterion, we deduce $I^\pm \subset \sigma_{ess}(H_0 + W)$. Consequently

$$\sigma_{ess}(H_0 + W) =]-\infty, -mc^2 + V^+] \cup [mc^2 + V^-, +\infty[.$$

Finally, using Weyl's theorem we obtain

$$\sigma_{ess}(H) = \sigma_{ess}(H_0 + W),$$

and the proposition holds. □

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